Illustration of Modified Laplace Transform “Kamal Transform” for Solving Some Ordinary Differential Equations with Variable Coefficients

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مستخلص:

بناءً على تحويلات لابلاس وسومودو قمنا بتعديل تحويل جديد، وهو تحويل كمال، الذي يمكنه حلّ بعض المعادلات التفاضليّة العاديّة ذات المعاملات المُتغيِّرة والمعادلات التي لا يمكن حلّها باستخدام تحويل سومودو. وقد تم تقديم ثلاثة أمثلة لتوضيح كفاءة تحويل كمال في حلّ هذا المعادلات.

Abstract:

Based on Laplace and Sumudu transforms we modified a new transform , Kamal transforms, which is capable for solving some ordinary differential equations with variable coefficients, equations that cannot be solved with Sumudu transform, three examples were introduced to illustrate the efficiency of Kamal transform in solving these equations.

Keywords: Kamal transform ;Sumudu transform, Ordinary Differential Equations, Laplace transform.

1.Introduction:

Kamal transform [1-4], was developed and derived from Laplace transform to overcome some difficulties faced Sumudu transform in solving some differential equations [1]. It has been shown here how Sumudu transform stuck when applied to some variable coefficients ordinary differential equations and Kamal transform succeed to solve them effectively, easily and accurately.

The ordinary differential equations (ODE) with variable coefficients appear in many areas of applied sciences .examples of these equations are Euler equation, Bessel equation, Legendre equation and Laguerre equation. Methods of solving Linear and nonlinear ODEs with variable coefficient are therefore of fundamental importance for understanding nature and interpreting behavior in many phenomena's .Many transforms were used to solve differential equations [8,9,11], Laplace transform [5,6],Fourier transform, Sumudu transform [6,7,10], Elzaki transform [5,14,16], ZZ transform[15,17], Natural transform [13], and Aboodh transform [5,12,15].These transforms faced some difficulties in solving ordinary differential equations with variable coefficients , some transforms can solve only simple cases in special situations. Ordinary differential equations have wide range of applications in many areas , in biology (spread of epidemics), medicine (growth of tambours), sociology (emigration rates), psychology (learning theories), economics (option pricing), chemistry (reaction rates), physics (dynamics of a laser) and engineering (electric circuits). Methods for solving differential equations are therefore of fundamental importance for understanding nature and technology. Purpose of this paper is to solve ordinary differential equations with variable coefficients which were not solved by Sumudu transform. Recently Abdelilah Kamal [1-4], introduced a new integral transform , named it as Kamal transform.

2. Fundamental Facts of Kamal and Sumudu Transforms:

In this section, we introduce the basic properties of single Kamal and Sumudu transforms.

2.1. Fundamental Facts of Single Kamal Transform: [1,3]

Definition 2.1. Let $f(t)$ be a function of t specified for $t > 0$.Then Kamal transform of $f(t)$, denoted by $K[f(t)]$, is defined by

$$K[f(t)] = \int_0^\infty f(t)e^{\frac{-t}{v}}dt = G(v) \qquad t \geq 0 \quad , \quad (1)$$

and the inverse of Kamal transform is given by

$$K^{-1}[G(v)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{t}{v}}G(v)dv = f(t) \quad , \quad t > 0\,. \qquad (2)$$

Theorem 2.1. (*Existence conditions*). If $f(t)$ is piecewise continuous function on the interval $[0,\infty)$ and of exponential order ϑ. Then $K[f(t)]$ exists for $Re(v) > \vartheta$ and satisfies

$$|f(t)| \leq \mu e^{\vartheta t}, \qquad (3)$$

where μ is positive constant. Then, the Kamal transform converges absolutely for $Re(v) > \vartheta$.

Proof: Using the definition of Kamal transform, we get

$$|G(v)| = \left|\int_0^\infty e^{-\frac{t}{v}}[f(t)]\,dt\right| \leq \int_0^\infty e^{-\frac{t}{v}}|f(t)|\,dt$$

$$\leq \mu\int_0^\infty e^{-\left(\frac{1}{v}-\vartheta\right)t}dt$$

$$= \frac{v\mu}{1-\vartheta v}, \qquad Re(v) > \vartheta.$$

Therefore, the Kamal transform converges absolutely for $Re(v) > \vartheta$.

2.2. Kamal Transform of the Some Functions:

In this section we find Kamal transform of simple functions. For any function $f(t)$, we assume that the integral Eq. (1) exist. The sufficient conditions for the existence of Kamal transform are that $f(t)$ for $t \geq 0$ be piecewise continuous and of exponential order, otherwise Kamal transform may or may not exist.

i. Let $f(t) = 1,\ t > 0$ .Then :

$$K[1] = \int_0^\infty e^{-\frac{t}{v}}[1]dt = -\frac{e^{-\frac{t}{v}}}{\frac{1}{v}}\Bigg|_{t=0}^{\infty} = v, \qquad Re(v) > 0.$$

ii. Let $f(t) = t\ \ , t > 0$. Then:

$$K[t] = \int_0^\infty e^{-\frac{t}{v}}[t]dt$$

Using integration by part , we have

$$K[t] = \int_0^\infty e^{-\frac{t}{v}}[t]dt = \left(-\frac{te^{-\frac{t}{v}}}{\frac{1}{v}}\right)_0^\infty - \int_0^\infty \frac{e^{-\frac{t}{v}}}{\frac{1}{v}}dt = -\frac{e^{-\frac{t}{v}}}{\frac{1}{v^2}}\Bigg|_{t=0}^{\infty}$$

$$= v^2.$$

In the general case if $n \geq 0$ is integer number, then.

$K[t^n] = n!\, v^{n+1}$.

iii. Let $f(t) = e^{\beta t}\ \ , t > 0$ and β is constant. Then:

$$K[e^{\beta t}] = \int_0^\infty e^{-\frac{t}{v}}[e^{\beta t}]dt = \int_0^\infty e^{-\left(\frac{t}{v}-\beta\right)t}dt = -\frac{e^{-\left(\frac{1}{v}-\beta\right)t}}{\frac{1}{v}-\beta}\Bigg|_{t=0}^{\infty}$$

$$= \frac{v}{1-v\beta}.$$

Similarly,

$$K[e^{i\beta t}] = \frac{v}{1-iv\beta}.$$

Using the property of complex analysis, we have:

$$K[e^{i\beta t}] = \frac{v-iv^2\beta}{1+v^2\beta^2}.$$

Using Euler's formulas: $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$, $\cos x = \frac{e^{ix}+e^{-ix}}{2}$,

and the formulas: $\sinh x = \frac{e^{x}-e^{-x}}{2}$, $\cosh x = \frac{e^{x}+e^{-x}}{2}$.

Therefore, we conclude the following:

$K[\sin(\beta t)] = \frac{v^2\beta}{1+v^2\beta^2}$,

$K[\cos(\beta t)] = \frac{v}{1+v^2\beta^2}$,

$K[\sinh(\beta t)] = \frac{v^2\beta}{1-v^2\beta^2}$,

$K[\cosh(\beta t)] = \frac{v}{1-v^2\beta^2}$.

Where $Im(\beta) < Re(v)$.

2.3. Some important theorems of Kamal Transform:

Theorem 2.2. (*Shifting Property*)**:** Let $f(t)$ be a continuous function and $K[f(t)] = G(v)$.Then:

$$K[e^{\beta t}f(t)] = G\left(\frac{v}{1-v\beta}\right). \quad (4)$$

Proof:

$$K[e^{\beta t}f(t)] = \int_0^\infty e^{-\frac{t}{v}}[e^{\beta t}f(t)]dt = \int_0^\infty e^{-\left(\frac{1}{v}-\beta\right)t}[f(t)]dt$$

$$= \int_0^\infty e^{-\left(\frac{1-v\beta}{v}\right)t}[f(t)]dt = G\left(\frac{v}{1-v\beta}\right).$$

Theorem 2.3. (*Periodic Function*)**:** Let $K[f(t)]$ exists, where $f(t)$ periodic function of period β such that:

$$f(t+\beta) = f(t)\,,\forall t.$$

Then:

$$K[f(t)] = \left(\frac{\int_0^p e^{-\frac{t}{v}}[f(t)]dt}{\left(1-e^{-\frac{\beta}{v}}\right)}\right). \qquad (5)$$

Proof: Using the definition of Kamal transform, we get:

$$K[f(t)] = \int_0^\infty e^{-\frac{t}{v}}[f(t)]dt. \qquad (6)$$

Using the property of improper integral, Eq.(6) can be written as:

$$K[f(t)] = \int_0^p e^{-\frac{t}{v}}[f(t)]dt + \int_p^\infty e^{-\frac{t}{v}}[f(t)]dt. \qquad (7)$$

Putting $t = \beta + \tau$ on the second integral in Eq.(7). We obtain

$$G(v) = \int_0^p e^{-\frac{t}{v}}[f(t)]dt$$
$$+ \int_0^\infty e^{-\frac{\beta+\tau}{v}}[f(\beta+\tau)]d\tau. \qquad (8)$$

Using the periodicity of the function $f(t)$, Eq.(8) can be written by:

$$G(v) = \int_0^p e^{-\frac{t}{v}}[f(t)]dt + e^{-\frac{\beta}{v}}\int_0^\infty e^{-\frac{\tau}{v}}[f(\tau)]d\tau. \qquad (9)$$

Using the definition of Kamal transform, we get:

$$G(v) = \int_0^p e^{-\frac{t}{v}}[f(t)]dt + e^{-\frac{\beta}{v}}G(v). \qquad (10)$$

Thus, Eq.(10) can be simplified into:

$$G(v) = \frac{1}{\left(1 - e^{-\frac{\beta}{v}}\right)}\left(\int_0^p e^{-\frac{t}{v}}[f(t)]dt\right).$$

Theorem 2.4. (*Heaviside Function*)**:** Let $K[f(t)]$ exists and $K[f(t)] = G(v)$, then:

$$K[f(t-\varepsilon)H(t-\varepsilon)] = e^{-\frac{\varepsilon}{v}}G(v). \qquad (11)$$

where $H(t-\varepsilon)$ is the Heaviside unit step function defined as:

$$H(t-\varepsilon) = \begin{cases} 1 & , t > \varepsilon , \\ 0 & , \quad \text{Otherwise.} \end{cases}$$

Proof: Using the definition of Kamal transform, we get:

$$K[f(t-\varepsilon)H(t-\varepsilon)] = \int_0^\infty e^{-\frac{t}{v}}[f(t-\varepsilon)H(t-\varepsilon)]dt$$

$$= \int_0^\infty e^{-\frac{t}{v}}[f(t-\varepsilon)]dt. \ (12)$$

Putting $t - \varepsilon = \tau$ in Eq.(12). We obtain

$$K[f(t-\varepsilon)H(t-\varepsilon)]$$

$$= \int_0^\infty e^{-\frac{\tau+\varepsilon}{v}}[f(\tau)]d\tau. \qquad (13)$$

Thus, Eq.(13) can be simplified into:

$$K[f(t-\varepsilon)H(t-\varepsilon)] = e^{-\frac{\varepsilon}{v}}\int_0^\infty e^{-\frac{\tau}{v}}[f(\tau)]d\tau$$

$$= e^{-\frac{\varepsilon}{v}}G(v). \qquad (14)$$

Theorem 2.5.(*Convolution Theorem*)**:** Let $K[f(t)]$ and $K[w(t)]$ exist and $K[f(t)] = G(v)$, $K[w(t)] = W(v)$, then:

$$K[f * w(t)] = G(v)W(v), \tag{15}$$

where $f * w(t) = \int_0^t f(\tau)w(t-\tau)\,d\tau$ and the symbol $*$ denotes the single convolution with respect to t.

Proof: Using the definition of Kamal transform, we get:

$$K[f * w(t)] = \int_0^\infty e^{-\frac{t}{v}}[f * w(t)]dt$$

$$= \int_0^\infty e^{-\frac{t}{v}}\left[\int_0^t f(\tau)w(t-\tau)\,d\tau\right]dt. \tag{16}$$

Using the Heaviside unit step function, Eq.(16) can be written as

$$K[f * w(t)] = \int_0^\infty \left(\int_0^\infty e^{-\frac{t}{v}} f(\tau)w(t-\tau)\,H(t-\tau)d\tau\right)dt. \tag{17}$$

Thus, Eq.(17) can be written as

$$K[f * w(t)] = \int_0^\infty f(\tau)d\tau \cdot \int_0^\infty e^{-\frac{t}{v}} w(t-\tau)\,H(t-\tau)dt$$

$$= W(v)\int_0^\infty e^{-\frac{\tau}{v}}[f(\tau)]d\tau = G(v)W(v).$$

Theorem 2.6.(*Derivatives Properties*)**:** Let $G(v)$ is the Kamal transform of $f(t)$ then:

i) $K[f'(t)] = \frac{1}{v}G(v) - f(0).$

ii) $K[f''(t)] = \frac{1}{v^2}G(v) - \frac{1}{v}f(0) - f'(0).$

iii) $K[f^{(n)}(t)] = v^{(-n)}G(v) - \sum_{k=0}^{n-1} v^{k-n+1} f^{(k)}(0).$

Proof: i) By the definition of Kamal transform, we have:

$$K[f'(t)] = \int_0^\infty f'(t)e^{\frac{-t}{v}}\,dt. \tag{18}$$

Integrating by parts, Eq. (18) can be written as

$$K[f'(t)] = \frac{1}{v}G(v) - f(0). \qquad (19)$$

ii)□□ $K\left[f''(t)\right] = \int_0^\infty f''(t)e^{\frac{-t}{v}}\,dt$

Integrating by parts, we get

$K[f''(t)] = v^{-2}G(v) - v^{-1}f(0) - f'(0)$

iii) Can be proof by mathematical induction .

Theorem 2.7. Let $G(v)$is Kamal transform of $f(t)$,then :

a) $K[tf(t)] = v^2\frac{d}{dv}(G(v))$

b) $K[tf'(t)] = v^2\frac{d}{dv}\left[\frac{1}{v}G(v) - f(0)\right]$

c) $K[tf''(t)] = v^2\frac{d}{dv}\left[\frac{1}{v^2}G(v) - \frac{1}{v}f(0) - f'(0)\right]$

d) $K[t^2f(t)] = v^4\frac{d^2}{dv^2}(G(v)) + 2v^3\frac{d}{dv}(G(v))$

e) $K[t^2f'(t)] = v^4\frac{d^2}{dv^2}\left(\frac{1}{v}G(v) - f(0)\right) + 2v^3\frac{d}{dv}\left(\frac{1}{v}G(v) - f(0)\right)$

f) $K[t^2f''(t)] = v^4\frac{d^2}{dv^2}\left(\frac{1}{v^2}G(v) - \frac{1}{v}f(0) - f'(0)\right) + 2v^3\frac{d}{dv}\left(\frac{1}{v^2}G(v) - \frac{1}{v}f(0) - f'(0)\right)$

Proof :i) By the definition , we have:

$$G(v) = \int_0^\infty f(t)e^{\frac{-t}{v}}\,dt$$

Differentiate two sides with respect to v :

$$\frac{d}{dv}(G(v))=\frac{\partial}{\partial v}\int_0^\infty e^{\frac{-t}{v}}f(t)dt, \qquad (20)$$

And ,

$$G'(v)=\int_0^\infty \frac{\partial}{\partial v}\left(e^{\frac{-t}{v}}\right)f(t)dt=\frac{1}{v^2}\int_0^\infty e^{\frac{-t}{v}}tf(t)dt=\frac{1}{v^2}K[tf(t)]. \qquad (21)$$

Thus ,

$$K[tf(t)]=v^2\frac{d}{dv}(G(v)). \qquad (22)$$

To prove (ii) and (iii) , put $f(t)=f'(t)$ and $f(t)=f''(t)$ respectively in Eq.(22) .

iv) By differentiating Eq.(20) with respect to v , we get:

$$\frac{d}{dv}(G'(v))=\frac{\partial}{\partial v}\left\{\frac{1}{v^2}\left(\int_0^\infty e^{\frac{-t}{v}}tf(t)dt\right)\right\}$$

And ,

$$G''(v)=\frac{1}{v^2}\int_0^\infty \frac{t^2}{v^2}f(t)e^{\frac{-t}{v}}dt-\frac{2}{v^3}\int_0^\infty tf(t)e^{\frac{-t}{v}}dt$$

Thus ,

$$K[t^2f(t)]=v^4\frac{d^2}{dv^2}(G(v))+2v^3\frac{d}{dv}(G(v)). \qquad (23)$$

To prove (v) and (vi) , put $f(t)=f'(t)$ and $f(t)=f''(t)$ respectively in Eq.(23) .

The previous results of Kamal transform to some basic functions, some theorems and basic derivatives are summed up in the Table below:

Table 1:
Kamal transform to some basic functions.

$f(t)$	$K[f(t)] = G(v)$
1	v
t	v^2
t^β	$\beta!\, v^{\beta+1}$
$e^{\beta t}$	$\frac{v}{1 - v\beta}$
$\sin(\beta t)$	$\frac{v^2\beta}{1 - v^2\beta^2}$
$\cos(\beta t)$	$\frac{v}{1 - v^2\beta^2}$
$\sinh(\beta t)$	$\frac{v^2\beta}{1 + v^2\beta^2}$
$\cosh(\beta t)$	$\frac{v}{1 + v^2\beta^2}$
$J_0(t)$	$\frac{v}{\sqrt{1 + v^2}}$
$e^{\beta t} f(t)$	$G\left(\frac{v}{1 - v\beta}\right)$
$f(t - \varepsilon)H(t - \varepsilon)$	$e^{-\frac{\varepsilon}{v}} G(v)$
$(f * w)(t)$	$G(v)W(v)$
$f'(t)$	$v^{-1}G(v) - f(0)$
$f''(t)$	$v^{-2}G(v) - \frac{1}{v} f(0) - f'(0)$

$f^{(n)}(t)$	$v^{-n}G(v)-\sum_{k=0}^{n-1} v^{k-n+1}f^{(k)}(0)$
$tf(t)$	$v^2\frac{d}{dv}(G(v))$
$tf'(t)$	$v^2\frac{d}{dv}\left[\frac{1}{v}G(v)-f(0)\right]$
$tf''(t)$	$v^2\frac{d}{dv}\left[\frac{1}{v^2}G(v)-\frac{1}{v}f(0)-f'(0)\right]$
$t^2f(t)$	$v^4\frac{d^2}{dv^2}(G(v))+2v^3\frac{d}{dv}(G(v))$
$t^2f'(t)$	$v^4\frac{d^2}{dv^2}\left(\frac{1}{v}G(v)-f(0)\right)+2v^3\frac{d}{dv}\left(\frac{1}{v}G(v)-\right.$
$t^2f''(t)$	$v^4\frac{d^2}{dv^2}\left(\frac{1}{v^2}G(v)-\frac{1}{v}f(0)-f'(0)\right)$ $+2v^3\frac{d}{dv}\left(\frac{1}{v^2}G(v)-\frac{1}{v}f(0)-\right.$

2.4. Fundamental Facts of single Sumudu transform: [5,6]

Definition 2.2. Let $f(t)$ be a function of t specified for $t>0$.Then Sumudu transform of $f(t)$, denoted by $S[f(t)]$, is defined by

$$S[f(t)]=\frac{1}{u}\int_0^\infty f(t)e^{\frac{-t}{u}}dt=F(u) \qquad t\geq 0 \quad , \tag{24}$$

and the inverse of Sumudu transform is given by

$$S^{-1}[F(u)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{\frac{t}{u}}}{u} F(u)du = f(t) \quad , \quad t > 0 . \qquad (25)$$

Theorem 2.1. (*Existence conditions*). If $f(t)$ is piecewise continuous function on the interval $[0,\infty)$ and of exponential order ϑ. Then $S[f(t)]$ exists for $Re(v) > \vartheta$ and satisfies

$$|f(t)| \leq Ke^{\vartheta t}, \qquad (26)$$

where K is positive constant. Then, the Sumudu transform converges absolutely for $Re(u) > \vartheta$.

Proof:

Using the definition of Sumudu transform, we get

$$|F(u)| = \left|\frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}[f(t)]\,dt\right| \leq \int_0^\infty \frac{1}{u}e^{-\frac{t}{u}}|f(t)|\,dt$$

$$\leq K\int_0^\infty \frac{1}{u}e^{-\left(\frac{1}{u}-\vartheta\right)t}dt$$

$$= \frac{K}{1-\vartheta u}, \qquad Re\left(\frac{1}{u}\right) > \vartheta.$$

Therefore, the Sumudu transform converges absolutely for $Re\left(\frac{1}{u}\right) > \vartheta$.

2.5. Sumudu Transform of the Some Functions:

In this section we find Sumudu transform of simple functions. For any function $f(t)$, we assume that the integral Eq. (24) exist. The sufficient conditions for the existence of Sumudu transform are that $f(t)$ for $t \geq 0$ be piecewise continuous and of exponential order, otherwise Sumudu transform may or may not exist.

i. Let $f(t) = 1,\ t > 0$ .Then :

$$S[1] = \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}[1]dt = -\frac{1}{u}\frac{e^{-\frac{t}{u}}}{\frac{1}{u}}\Bigg|_{t=0}^{\infty} = 1.$$

ii. Let $f(t) = t\ \ ,t > 0$. Then:

$$S[t] = \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}[t]dt.$$

Using integration by part , we have

$$S[t] = \frac{1}{u}\int_0^\infty e^{-\frac{t}{v}}[t]dt$$

$$= \frac{1}{u}\left(-\frac{te^{-\frac{t}{u}}}{\frac{1}{u}}\right)_0^\infty - \frac{1}{u}\int_0^\infty \frac{e^{-\frac{t}{u}}}{\frac{1}{u}}dt = -\frac{1}{u}\frac{e^{-\frac{t}{u}}}{\frac{1}{u^2}}\Bigg|_{t=0}^{\infty}$$

$$= u.$$

In the general case if $n \geq 0$ is integer number, then.

$$S[t^n] = n!\,u^n.$$

iii. Let $f(t) = e^{\beta t}\ ,t > 0$ and β is constant. Then:

$$S[e^{\beta t}] = \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}[e^{\beta t}]dt = \frac{1}{u}\int_0^\infty e^{-\left(\frac{t}{u}-\beta\right)t}dt$$

$$= -\frac{1}{u}\frac{e^{-\left(\frac{1}{u}-\beta\right)t}}{\frac{1}{u}-\beta}\Bigg|_{t=0}^{\infty} = \frac{1}{1-u\beta}.$$

Similarly,

$$S[e^{i\beta t}] = \frac{1}{1-iu\beta}.$$

Using the property of complex analysis, we have:

$$K[e^{i\beta t}] = \frac{1 + iu\beta}{1 + u^2\beta^2}.$$

Using Euler's formulas: $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$, $\cos x = \frac{e^{ix} + e^{-ix}}{2}$,

and the formulas: $\sinh x = \frac{e^{x} - e^{-x}}{2}$, $\cosh x = \frac{e^{x} + e^{-x}}{2}$.

Therefore, we conclude the following:

$K[\sin(\beta t)] = \frac{u\beta}{1+u^2\beta^2}$,

$K[\cos(\beta t)] = \frac{1}{1+u^2\beta^2}$,

$K[\sinh(\beta t)] = \frac{u\beta}{1-u^2\beta^2}$,

$K[\cosh(\beta t)] = \frac{1}{1-u^2\beta^2}$.

Where $Im(\beta) < Re\left(\frac{1}{u}\right)$.

2.6. Some important theorems of Sumudu Transform:

Theorem 2.8. (*Shifting Property*)**:** Let $f(t)$ be a continuous function and $S[f(t)] = F(u)$.Then:

$$S[e^{\beta t} f(t)] = G\left(\frac{u}{1-u\beta}\right). \quad (27)$$

Proof:

$$S[e^{\beta t} f(t)] = \frac{1}{u}\int_0^{\infty} e^{-\frac{t}{u}}[e^{\beta t} f(t)]dt = \frac{1}{u}\int_0^{\infty} e^{-\left(\frac{1}{u}-\beta\right)t}[f(t)]dt$$

$$= \frac{1}{u}\int_0^{\infty} e^{-\left(\frac{1-v\beta}{u}\right)t}[f(t)]dt = G\left(\frac{u}{1-u\beta}\right).$$

Theorem 2.9. (*Periodic Function*)**:** Let $S[f(t)]$ exists, where $f(t)$ periodic function of period β such that:

$$f(t+\beta) = f(t), \forall t.$$

Then:

$$S[f(t)] = \left(\frac{\frac{1}{u}\int_0^p e^{-\frac{t}{u}}[f(t)]dt}{\left(1 - e^{-\frac{\beta}{u}}\right)} \right). \qquad (28)$$

Proof:

Using the definition of Sumudu transform, we get:

$$S[f(t)] = \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}[f(t)]dt. \qquad (29)$$

Using the property of improper integral, Eq.(29) can be written as:

$$S[f(t)] = \frac{1}{u}\int_0^p e^{-\frac{t}{u}}[f(t)]dt + \frac{1}{u}\int_p^\infty e^{-\frac{t}{u}}[f(t)]dt. \qquad (30)$$

Putting $t = \beta + \tau$ on the second integral in Eq.(30). We obtain

$$F(u) = \frac{1}{u}\int_0^p e^{-\frac{t}{u}}[f(t)]dt + \frac{1}{u}\int_0^\infty e^{-\frac{\beta+\tau}{u}}[f(\beta + \tau)]d\tau. \qquad (31)$$

Using the periodicity of the function $f(t)$, Eq.(31) can be written by:

$$F(u) = \frac{1}{u}\int_0^p e^{-\frac{t}{u}}[f(t)]dt + e^{-\frac{\beta}{u}}\left(\frac{1}{u}\int_0^\infty e^{-\frac{\tau}{u}}[f(\tau)]d\tau\right). \qquad (32)$$

Using the definition of Sumudu transform, we get:

$$F(u) = \frac{1}{u}\int_0^p e^{-\frac{t}{u}}[f(t)]dt + e^{-\frac{\beta}{u}}F(u). \qquad (33)$$

Thus, Eq.(33) can be simplified into:

$$F(u) = \frac{1}{\left(1 - e^{-\frac{\beta}{u}}\right)}\left(\frac{1}{u}\int_0^p e^{-\frac{t}{u}}[f(t)]dt\right).$$

Theorem 2.10. (*Heaviside Function*)**:** Let $S[f(t)]$ exists and $S[f(t)] = F(u)$, then:

$$S[f(t-\varepsilon)H(t-\varepsilon)] = e^{-\frac{\varepsilon}{u}}F(u). \qquad (34)$$

where $H(t-\varepsilon)$ is the Heaviside unit step function defined as:

$$H(t-\varepsilon) = \begin{cases} 1 & , t > \varepsilon , \\ 0 & , \quad \text{Otherwise.} \end{cases}$$

Proof:

Using the definition of Sumudu transform, we get:

$$S[f(t-\varepsilon)H(t-\varepsilon)] = \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}[f(t-\varepsilon)H(t-\varepsilon)]dt$$

$$= \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}[f(t-\varepsilon)]dt. \qquad (35)$$

Putting $t - \varepsilon = \tau$ in Eq.(35). We obtain

$$S[f(t-\varepsilon)H(t-\varepsilon)] = \frac{1}{u}\int_0^\infty e^{-\frac{\tau+\varepsilon}{u}}[f(\tau)]d\tau. \qquad (36)$$

Thus, Eq.(36) can be simplified into:

$$S[f(t-\varepsilon)H(t-\varepsilon)] = \frac{1}{u}e^{-\frac{\varepsilon}{u}}\int_0^\infty e^{-\frac{\tau}{u}}[f(\tau)]d\tau = e^{-\frac{\varepsilon}{u}}F(u).$$

Theorem 2.11.(*Convolution Theorem*)**:** Let $S[f(t)]$ and $S[w(t)]$ exist and $S[f(t)] = F(u)$, $S[w(t)] = W(u)$, then:

$$S[f * w(t)] = uF(u)W(u). \quad (37)$$

where $f * w(t) = \int_0^t f(\tau)w(t-\tau)\,d\tau$ and the symbol $*$ denotes the single convolution with respect to t.

Proof: Using the definition of Sumudu transform, we get:

$$K[f * w(t)] = \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}[f * w(t)]dt$$

$$= \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}\left[\int_0^t f(\tau)w(t-\tau)\,d\tau\right]dt. \quad (38)$$

Using the Heaviside unit step function, Eq.(38) can be written as

$$K[f * w(t)] = \frac{1}{u}\int_0^\infty \left(\int_0^\infty e^{-\frac{t}{u}} f(\tau)w(t-\tau)H(t-\tau)d\tau\right)dt. \quad (39)$$

Thus, Eq.(39) can be written as

$$K[f * w(t)] = \frac{1}{u}\int_0^\infty f(\tau)d\tau \cdot \int_0^\infty e^{-\frac{t}{u}} w(t-\tau)H(t-\tau)dt$$

$$= W(v)\int_0^\infty e^{-\frac{\tau}{u}}[f(\tau)]d\tau = uF(u)W(u).$$

Theorem 2.12.(*Derivatives Properties***):** Let $F(u)$ is the Sumudu transform of $f(t)$ then:

i. $S[f'(t)] = \frac{1}{u}F(u) - \frac{1}{u}f(0).$

ii. $S[f''(t)] = \frac{1}{u^2}F(u) - \frac{1}{u^2}f(0) - \frac{1}{u}f'(0).$

iii. $S[tf'(t)] = u^2\frac{d}{du}\left[\frac{F(u)-f(0)}{u}\right].$

iv. $S[tf''(t)] = u^2\frac{d}{du}\left[\frac{F(u)-f(0)-f'(0)}{u^2}\right] + u\left[\frac{F(u)-f(0)-f'(0)}{u^2}\right].$

$$S[t^2 f''(t)] = u^4 \frac{d^2}{dv^2}\left(\frac{F(u)-f(0)-f\prime(0)}{u^2}\right) +$$

v. $$4u^3 \frac{d}{du}\left(\frac{F(u)-f(0)-f\prime(0)}{u^2}\right) + 2u^2\left(\frac{F(u)-f(0)-f\prime(0)}{u^2}\right).$$

3.Methodology

In this part we interduce the general methodology for applying the Kamal and Sumudu transforms to solve the ordinary differential equations defined in this section:

- Apply Sumudu transform to the main equation.
- Solve the algebraic equation resulting from the effect of Sumudu transform on the main equation.
- Obtain the final solution using the inverse Sumudu transform method.
- Apply Kamal transform to the main equation.
- Solve the algebraic equation resulting from the effect of Kamal transform on the main equation.
- Obtain the final solution using the inverse Kamal transform method.
- Finally, we compare the solutions.

4.Applications:

In this section, we introduce three interesting examples of ODEs and to solve them by the current method.

Example 4.1:

Let's consider the following ordinary differential equation :

$$t^2 y'' + 4ty' + 2y = 12t^2 , y'(0) = y(0) = 0. \qquad (40)$$

Use Sumudu transform :

Sumudu transform to Eq.(40) is:

$u^2 F''(u) + 4uF'(u) + 2F(u) = 24u^2$

Which is the same Eq. (40), this means that Sumudu transform cannot solve this equation.

Use Kamal transform :

Applying Kamal transform to both sides of Eq.(40)

$$K[t^2y'' + 4ty' + 2y] = K[12t^2].$$

(41)

Using the differential property of Kamal transform , Eq.(41) can be written as:

$$\left\{v^4 \frac{d^2}{dv^2}\left(\frac{1}{v^2}K(y) - \frac{1}{v}y(0) - y'(0)\right) + 2v^3 \frac{d}{dv}\left(\frac{1}{v^2}K(y) - \frac{1}{v}y(0) - y'(0)\right)\right\} + \left\{4v^2 \frac{d}{dv}\left(\frac{1}{v}K(y) - y(0)\right)\right\} + 2K(y) = 24v^3$$

Now , applying the initial conditions , we get

$$\{v^2K''(y) - 2vK'(y) + 2K(y)\} + 4vK'(y) - 4K(y) + 2K(y) = 24v^3$$

And ,$v^2K''(y) + 2vK'(y) = 24v^3$

or

$$\frac{d}{dv}\left(v^2K'(y)\right) = 24v^3,$$

And ,

$$K'(y) = \frac{1}{v^2}\left(\int 24v^3\,dv + c_1\right).$$

Thus ,

$$K(y) = 2v^3 + \frac{c_1}{v} + c_2. \qquad (42)$$

Now , applying inverse Kamal transform to Eq.(42) , then the solution of Eq.(40) is:

$$y(t) = K^{-1}\left[2v^3 + \frac{c_1}{v} + c_2\right] = t^2 \text{ , } where \quad c_1 = c_2 = 0.$$

Example 4.2:

Let's consider the following ordinary differential equation :

$$t^2y''' + 6ty'' + 6y' = 60t^2 \quad ,$$

$$y'''(0) = y''(0) = y(0) = 0. \qquad (43)$$

Use Sumudu transform :

Sumudu transform to Eq.(43) is:

$u^2F''(u) + 4uF'(u) + 2F(u) = 120u^2.$

Which is the same Eq. (40), this means that Sumudu transform cannot solve this equation.

Use Kamal transform :

Applying Kamal transform to both sides of Eq.(43)

$$K[t^2y''' + 6ty'' + 6y' =] = K[60t^2]. \qquad (44)$$

Using the differential property of Kamal transform and applying the initial conditions , Eq.(44) can be written as:

$$\left\{v^2K''(y) - 4vK'(y) + \frac{6}{v}K(y)\right\} + \left\{6vK'(y) - \frac{12}{v}K(y)\right\} + \frac{6}{v}K(y) = 120v^3,$$

And , $v^2K''(y) + 2vK'(y) = 120v^4.$

or

$$\frac{d}{dv}\left(v^2K'(y)\right) = 120v^4.$$

And ,

$$K'(y) = \frac{1}{v^2}\left(\int 120v^4\,dv + c_1\right).$$

Thus ,

$$K(y) = 6v^4 + \frac{c_1}{v} + c_2. \qquad (45)$$

Now , applying inverse Kamal transform to Eq.(45) , then the solution of Eq.(43) is:

$y(t) = K^{-1}\left[6v^4 + \frac{c_1}{v} + c_2\right] = t^3$, *where* $c_1 = c_2 = 0.$

Example 4.3:

Let's consider the following ordinary differential equation :

$$t^2y''' - 6y' = 0 \quad , \quad y'''(0) = y''(0) = y(0) = 0. \quad (45)$$

Use Sumudu transform :

Sumudu transform to Eq.(45) is:

$u^2F''(u) - 2u\,F'(u) - 4F(u) = 0.$

As we previously showed in equation (40) , this equation also can not be solved by Sumudu.

Use Kamal transform :

Applying Kamal transform to both sides of Eq.(45)

$$K[t^2y''' - 6y' =] = 0. \quad (46)$$

Using the differential property of Kamal transform and applying the initial conditions , Eq.(46) can be written as:

$$\left\{v^2K''(y) - 4vK'(y) + \frac{6}{v}K(y)\right\} - \frac{6}{v}K(y) = 0.$$

Thus, after simplifying, we have $\frac{K''(y)}{K'(y)} = \frac{4}{v}$, afterintegration from both sides, so we have :

$\ln K'(y) = 4\ln v + \ln C \quad , K'(y) = cv^4.$

and hence $K(y) = \frac{C}{5}v^5 + C_1$. If $C_1 = 0$. Thus , $K(y) = \frac{C}{5}v^5$. By using the inverse of Kamal transform well have :

$$y(x) = \frac{C}{120}t^4.$$

5.Conclusion:

In this paper, Kamal Transform was introduced and applied to ordinary differential equations with variable coefficients .Illustration showed that the presented equations was not solved by Sumudu transform and solved by Kamal transform. One can conclude that Kamal transform is a very effective method for solving ordinary differential equations with variable coefficients , compared with Sumudu transform. In a large domain the accurate convergence of Kamal transform will be discussed in the coming research.

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